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Martingale representation for Poisson processes with applications to minimal variance hedging

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Abstract

We consider a Poisson process η on a measurable space equipped with a strict partial ordering, assumed to be total almost everywhere with respect to the intensity measure λ of η. We give a Clark–Ocone type formula providing an explicit representation of square integrable martingales (defined with respect to the natural filtration associated with η), which was previously known only in the special case, when λ is the product of Lebesgue measure on $\mathbb{R}_+$ and a σ-finite measure on another space $\mathbb{X}$. Our proof is new and based on only a few basic properties of Poisson processes and stochastic integrals. We also consider the more general case of an independent random measure in the sense of Itô of pure jump type and show that the Clark–Ocone type representation leads to an explicit version of the Kunita–Watanabe decomposition of square integrable martingales. We also find the explicit minimal variance hedge in a quite general financial market driven by an independent random measure.

MSC: primary 60G55; 60G44; secondary 60G51

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1. Introduction

Any square integrable martingale with respect to a Brownian filtration can be written as a stochastic integral; see [10] and Theorem 18.10 in [16]. This *martingale representation theorem*

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is an important result of stochastic analysis. Similar results are available for marked point processes (see e.g. [19,13] and the references given there) and for general semimartingales, see Section III.4 in [13]. For some Brownian martingales Clark [3] found a more explicit version of the integrand in the representation. Ocone [24] revealed the relationship of Clark's formula to Malliavin calculus.

The present paper is concerned with similar representations in the *Poisson* setting. More precisely, we give a Clark–Ocone type martingale representation formula when the underlying filtration is generated by a Poisson process η on a measurable space $(\mathbb{Y}, \mathcal{Y})$ equipped with a strict partial ordering. We assume that this ordering is total almost everywhere with respect to the *intensity measure* λ of η; see (2.1). Our main result (Theorem 2.1) provides a representation of square integrable martingales as a (stochastic) Kabanov–Skorohod integral with respect to the compensated Poisson process. If $\mathbb{Y} = \mathbb{R}_+ \times \mathbb{X}$, that is $\mathbb{Y}$ is the product of $\mathbb{R}_+ := [0, \infty)$ and a Borel space $\mathbb{X}$, special cases of this formula are well known. Stationary Poisson processes on $\mathbb{R}_+$ were treated in Picard [27], while [1] considered the more general case of a finite set $\mathbb{X}$. In [31], it was shown how to use the Malliavin calculus for Poisson processes developed in [25,14,23] and the results in [4] to get the Clark–Ocone formula under an additional integrability assumption in the case where the intensity measure λ of η is the product of Lebesgue measure and a σ-finite measure on $\mathbb{X}$. This is also the approach taken in [21,7] when treating pure jump Lévy processes (without referring to [31]). Translated to our setting, this is again the special case where λ has product form. Our proof of Theorem 2.1 is based on the explicit Fock space representation of Poisson functionals [20, Theorem 1.5] and the basic isometry properties of stochastic integrals, and is distinct from the proofs of related results that we have seen in the literature. In particular, we are not using any other martingale representation theorem for Poisson spaces. The integrand in the representation of Theorem 2.1 is *predictable* in the sense made precise by (2.5) below. We conclude Section 2 by providing a more direct approach to the Kabanov–Skorohod integral of predictable functions; see Theorem 2.6.

In Section 3, we discuss Theorem 2.1 in the case $\mathbb{Y} = \mathbb{R}_+ \times \mathbb{X}$, where $(\mathbb{X}, \mathcal{X})$ is a Borel space and $(s, x) < (s', x')$ if and only if $s < s'$. Our assumption on λ then means that

$$\lambda(\{t\} \times \mathbb{X}) = 0, \quad t \geq 0. \tag{1.1}$$

We do not assume λ to be of product form. We shall also show in Proposition 3.3 that our notion of a predictable function is essentially the same as the standard notion of a predictable function in stochastic analysis. Theorem 3.5 shows that the Kabanov–Skorohod integral of a predictable function coincides with the (standard) stochastic integral. This extends results in [14,23] for Poisson processes on $\mathbb{R}_+$.

In Section 4, we extend the setting of Section 3 and consider instead of the compensated Poisson process a more general centered independent random measure ζ (in the sense of [11]) on $\mathbb{R}_+ \times \mathbb{X}$. We assume that ζ has no Gaussian part and a σ-finite variance measure with diffuse projection onto the first coordinate. Then ζ can be represented in terms of a Poisson process η as above on $\mathbb{Y} := \mathbb{R}_+ \times \mathbb{X} \times (\mathbb{R} \setminus \{0\})$. Consequently, we can apply our Clark–Ocone type formula to obtain an explicit formula for the orthogonal projection of a square integrable function of η onto the space of all stochastic integrals against ζ; see Theorem 4.1. Such projections were first considered by Kunita and Watanabe [18] in the setting of continuous martingales. Later these ideas were extended to semimartingales; see e.g. Schweizer [28]. Using a different approach (and allowing for a Gaussian component) Di Nunno [5] proved a version of Theorem 4.1 for special ("core") functions of η. In fact, we prove our results in the more general case of an independent random measure ζ on a Borel space $(\mathbb{Y}', \mathcal{Y}')$ with a diffuse and σ-finite variance measure β such that $\mathbb{Y}'$ is totally ordered almost everywhere with respect to β.

In the final Section 5 we consider a quite general financial market with a continuum of assets, driven by an independent random measure without Gaussian component. Again all processes can be represented in terms of a Poisson process η on a suitable state space. A square integrable function $f(\eta)$ can then be interpreted as a contingent claim. Minimizing the L^2-distance between $f(\eta) - \mathbb{E}f(\eta)$ and a certain space of stochastic integrals against the assets, yields the *minimal variance hedge* of $f(\eta)$. Theorem 5.4 finds this hedge explicitly, while Theorem 5.5 identifies the claims that can be perfectly hedged. These theorems extend the main results in [2], which treats the case of a market driven by a finite number of independent square integrable Lévy processes.

2. Representation of Poisson martingales

Throughout the paper we consider a Poisson process η on a measurable space $(\mathbb{Y}, \mathcal{Y})$ with σ-finite intensity measure λ. The underlying probability space is denoted by $(\Omega, \mathcal{F}, \mathbb{P})$. We can interpret η as a random element in the space $\mathbf{N} := \mathbf{N}(\mathbb{Y})$ of σ-finite integer-valued measures μ on $\mathbb{Y}$ equipped with the smallest σ-field $\mathcal{N}$ making the mappings $\mu \mapsto \mu(B)$ measurable for all $B \in \mathcal{Y}$; see [16, p. 226]. We denote the distribution of η by $\mathbb{P}_\eta := \mathbb{P}(\eta \in \cdot)$.

We assume that $\mathbb{Y}$ is equipped with a transitive binary relation $<$ such that $\{(y, z) : y < z\}$ is a measurable subset of $\mathbb{Y}^2$ and such that $y < y$ fails for all $y \in \mathbb{Y}$. We also assume that $<$ totally orders the points of $\mathbb{Y}$ λ-a.e., that is

$$\lambda([y]) = 0, \quad y \in \mathbb{Y}, \tag{2.1}$$

where $[y] := \mathbb{Y} \setminus \{z \in \mathbb{Y} : z < y \text{ or } y < z\}$. Note that $y \in [y]$ for all $y \in \mathbb{Y}$. If, for instance, $<$ is a strict total order (such as the strict lexicographic order on $\mathbb{R}^d$), then $[y] = \{y\}$ and (2.1) means that λ is diffuse. However, our main example is the ordering defined before (1.1). For any $\mu \in \mathbf{N}$ let μ_y denote the restriction of μ to $y_\downarrow := \{z \in \mathbb{Y} : z < y\}$. Our final assumption on $<$ is that $(\mu, y) \mapsto \mu_y$ is a measurable mapping from $\mathbf{N} \times \mathbb{Y}$ to $\mathbf{N}$.

For $y \in \mathbb{Y}$ the *difference operator* D_y is given as follows. For any measurable $f : \mathbf{N} \to \mathbb{R}$ the function $D_y f$ on $\mathbf{N}$ is defined by

$$D_y f(\mu) := f(\mu + \delta_y) - f(\mu), \quad \mu \in \mathbf{N}, \tag{2.2}$$

where δ_y is the Dirac measure located at a point $y \in \mathbb{Y}$. This operator (also known as *add one cost operator*) occurs rather frequently in point process theory. Of particular importance for us is its interpretation as *Malliavin derivative* on Poisson spaces [12,23]; see [20] for further discussion. We need a version of the conditional expectation $\mathbb{E}[D_y f(\eta)|\eta_y]$ that is jointly measurable in all arguments. Thanks to the independence properties of a Poisson process we can and will work with

$$\mathbb{E}[D_y f(\eta)|\eta_y] := h(\eta, y), \tag{2.3}$$

where $h : \mathbf{N} \times \mathbb{Y} \to \mathbb{R}$ is defined by

$$h(\mu, y) := \mathbb{E}D_y f(\mu_y + \eta - \eta_y) \tag{2.4}$$

if this expectation is finite, and $h(\mu, y) := 0$ otherwise. Since $(\mu, y) \mapsto \mu_y$ is assumed to be measurable, the function h is measurable as well. Moreover, it satisfies

$$h(\mu, y) = h(\mu_y, y), \quad (\mu, y) \in \mathbf{N} \times \mathbb{Y}. \tag{2.5}$$

Justified by Proposition 3.3 we call a measurable function h with property (2.5) *predictable*; see Remark 3.6. This notion depends on the ordering $<$. The fact that this dependence is not reflected in our terminology, will not lead to confusion.

If $h : \mathbf{N} \times \mathbb{Y} \to \mathbb{R}$ is a measurable function then we denote by

$$\delta(h) \equiv \int h(\eta, y)\hat{\eta}(\mathrm{d}y)$$

the stochastic *Kabanov–Skorohod integral* of h with respect to the compensated Poisson process $\hat{\eta} := \eta - \lambda$ [14,29,15]. This integral is well defined only if h satisfies the integrability condition (2.18) below, and we shall recall its definition at (2.17). If $h \in L^2(\mathbb{P}_\eta \otimes \lambda)$ is predictable (i.e. (2.5) holds), then $\delta(h)$ is well defined; see Proposition 2.4. At the end of this section we will prove that the restriction of δ to the space of predictable elements of $L^2(\mathbb{P}_\eta \otimes \lambda)$ is the unique linear operator from this space to $L^2(\mathbb{P})$, satisfying the pathwise identity

$$\delta(h) = \int h(\eta, y)\eta(\mathrm{d}y) - \int h(\eta, y)\lambda(\mathrm{d}y) \quad \mathbb{P}\text{-a.s.} \tag{2.6}$$

for predictable $h \in L^1(\mathbb{P}_\eta \otimes \lambda) \cap L^2(\mathbb{P}_\eta \otimes \lambda)$ and the isometry relation

$$\mathbb{E}\delta(h)^2 = \mathbb{E} \int h(\eta, y)^2 \lambda(\mathrm{d}y). \tag{2.7}$$

If $h \in L^2(\mathbb{P}_\eta \otimes \lambda)$ is predictable and $A \in \mathcal{Y}$, then $\mathbf{1}_{\mathbf{N}\times A} h$ is also predictable, and we can define

$$\int_A h(\eta, y)\hat{\eta}(\mathrm{d}y) := \delta(\mathbf{1}_{\mathbf{N}\times A} h).$$

For $f \in L^2(\mathbb{P}_\eta)$ (i.e. for measurable $f : \mathbf{N} \to \mathbb{R}$ with $\mathbb{E} f(\eta)^2 < \infty$) we have the following representation of $f(\eta)$.

Theorem 2.1. *Let η be a Poisson process on $\mathbb{Y}$ with an intensity measure λ satisfying (2.1) and let $f \in L^2(\mathbb{P}_\eta)$. Then*

$$\mathbb{E} \int \mathbb{E}[D_y f(\eta)|\eta_y]^2 \lambda(\mathrm{d}y) < \infty \tag{2.8}$$

and we have $\mathbb{P}$-a.s. that

$$f(\eta) = \mathbb{E} f(\eta) + \int \mathbb{E}[D_y f(\eta)|\eta_y]\hat{\eta}(\mathrm{d}y). \tag{2.9}$$

Moreover, we have for any $y \in \mathbb{Y}$ that $\mathbb{P}$-a.s.

$$\mathbb{E}[f(\eta)|\eta_y] = \mathbb{E} f(\eta) + \int_{y\downarrow} \mathbb{E}[D_z f(\eta)|\eta_z]\hat{\eta}(\mathrm{d}z). \tag{2.10}$$

Define $M_y := \mathbb{E}[f(\eta)|\eta_y]$, $y \in \mathbb{Y}$, where f is as in (2.10). If $z < y$ then the σ-field $\sigma(\eta_z) := \eta_z^{-1}(\mathcal{N})$ is contained in $\sigma(\eta_y)$ and we have the martingale property $\mathbb{E}[M_y|\eta_z] = M_z$ a.s. Eq. (2.10) provides an explicit representation of the martingale (M_y) as stochastic integral of an explicitly known integrand.

Remark 2.2. While (2.6) provides an appealing pathwise definition of δ in the predictable case, in other cases it is not applicable and can be misleading. If, for instance, λ is Lebesgue measure on $[0, 1]$, $h(\eta, t) = \eta([0, t])$ and $\tilde{h}(\eta, t) = \eta([0, t))$, then $\tilde{h}$ is predictable but h is not. By isometry $\delta(h) = \tilde{\delta}(h)$ but

$$\int h(\eta, t)\eta(\mathrm{d}t) \neq \int \tilde{h}(\eta, t)\eta(\mathrm{d}t).$$

So (2.6) is invalid in this case. A pathwise representation of δ for general integrands (satisfying some integrability assumption) is given in [20, Theorem 3.5].

Before proving Theorem 2.1, we state some definitions and preliminary observations. Let $f : \mathbf{N} \to \mathbb{R}$ be a measurable function. For $n \geq 2$ and $(y_1, \ldots, y_n) \in \mathbb{Y}^n$ we define a function $D^n_{y_1,\ldots,y_n} f : \mathbf{N} \to \mathbb{R}$ by an iterated application of the difference operator D, that is, inductively by

$$D^n_{y_1,\ldots,y_n} f := D^1_{y_1} D^{n-1}_{y_2,\ldots,y_n} f, \tag{2.11}$$

where $D^1_y := D_y$ and $D^0 f := f$. For $f \in L^2(\mathbb{P}_\eta)$ it was proved in [20] that $D^n_{y_1,\ldots,y_n} f(\eta)$ is integrable for λ^n-a.e. $(y_1, \ldots, y_n)$ and that

$$T_n f(y_1, \ldots, y_n) := \mathbb{E} D^n_{y_1,\ldots,y_n} f(\eta), \quad (y_1, \ldots, y_n) \in \mathbb{Y}^n, \tag{2.12}$$

defines a symmetric function in $L^2(\lambda^n)$. Moreover, we have the Wiener–Itô chaos expansion

$$f(\eta) = \sum_{n=0}^{\infty} \frac{1}{n!} I_n(T_n f), \tag{2.13}$$

where the series converges in $L^2(\mathbb{P})$; see again [20]. Here $I_n(g)$ denotes the nth *multiple Wiener–Itô integral* of a symmetric $g \in L^2(\lambda^n)$; see [11]. These integrals satisfy the orthogonality relations

$$\mathbb{E} I_m(g) I_n(h) = \mathbf{1}\{m = n\} m! \langle g, h\rangle_n \quad m, n \in \mathbb{N}_0, \tag{2.14}$$

where $\langle \cdot, \cdot \rangle_n$ denotes the scalar product in $L^2(\lambda^n)$ and $\mathbb{N}_0 := \{0, 1, 2, \ldots\}$. For a homogeneous Poisson process on the real line, the explicit chaos expansion (2.13) was proved in [12]. Stroock [30] has proved the counterpart of (2.13) for Brownian motion. Stroock's formula involves iterated Malliavin derivatives and requires stronger integrability assumptions on $f(\eta)$.

Let $h \in L^2(\mathbb{P}_\eta \otimes \lambda)$. Then $h(\cdot, y) \in L^2(\mathbb{P}_\eta)$ for λ-a.e. y and we may consider the chaos expansion

$$h(\eta, y) = \sum_{n=0}^{\infty} I_n(h_n(y)), \tag{2.15}$$

where $h_n(y) \in L^2(\lambda^n)$, $n \in \mathbb{N}_0$, are given by

$$h_n(y)(y_1, \ldots, y_n) := \mathbb{E} D^n_{y_1,\ldots,y_n} h(\eta, y). \tag{2.16}$$

Let $\tilde{h}_n$ be the *symmetrization* of this function, that is

$$\tilde{h}_n(y_1, \ldots, y_{n+1}) = \frac{1}{n+1} \sum_{i=1}^{n} h_n(y_i)(y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_{n+1}).$$

From (2.14) and (2.15) we obtain that $\tilde{h}_n \in L^2(\lambda^{n+1})$ and we can define the Kabanov–Skorohod integral [8,14,29,15,20] of h, denoted $\delta(h)$, by

$$\delta(h) := \sum_{n=0}^{\infty} I_{n+1}(\tilde{h}_n), \tag{2.17}$$

which converges in $L^2(\mathbb{P})$ provided that

$$\sum_{n=0}^{\infty}(n+1)! \int \tilde{h}_n^2 \mathrm{d}\lambda^{n+1} < \infty. \tag{2.18}$$

We need the following integration by parts formula from [23]; see also Proposition 3.4 in [20]. This formula shows that δ might be considered as the adjoint operator of D. We let $\|\cdot\|_n$ denote the norm in $L^2(\lambda^n)$.

Proposition 2.3. *Assume that $g \in L^2(\mathbb{P}_\eta)$ satisfies*

$$\sum_{n=1}^{\infty} \frac{1}{(n-1)!} \|T_n g\|_n^2 < \infty. \tag{2.19}$$

Let $h \in L^2(\mathbb{P}_\eta \otimes \lambda)$ with a chaos expansion satisfying (2.18). Then $\mathbb{E}\int (D_y g(\eta))^2 \lambda(\mathrm{d}y) < \infty$ and

$$\mathbb{E} \int D_y g(\eta) h(\eta, y) \lambda(\mathrm{d}y) = \mathbb{E} g(\eta) \delta(h). \tag{2.20}$$

Proposition 2.3 easily shows that δ is *closed*; see [14,23]. This means that if $h_k \in L^2(\mathbb{P}_\eta \otimes \lambda)$, $k \in \mathbb{N}$, satisfy (2.18), $h_k \to h$ in $L^2(\mathbb{P}_\eta \otimes \lambda)$ and $\delta(h_k) \to X$ in $L^2(\mathbb{P})$, then h satisfies (2.18) and $\delta(h) = X$ a.s. We shall use this fact repeatedly in what follows. Note in particular that $\delta(h) = 0$ a.s. if (and in fact only if) $h = 0$ $\mathbb{P}_\eta \otimes \lambda$-a.e.

The next result shows that the Kabanov–Skorohod integral of a predictable h is defined, if h is square integrable with respect to $\mathbb{P}_\eta \otimes \lambda$.

Proposition 2.4. *Let $h \in L^2(\mathbb{P}_\eta \otimes \lambda)$ be predictable. Then (2.18) holds.*

Proof. Consider the functions defined by (2.16). We claim that $h_n(y)(y_1, \dots, y_n) = 0$ whenever $y < y_i$ for some $i \in \{1, \dots, n\}$. Let us assume for instance that $y < y_n$. Since h is predictable, we have that

$$D_{y_n} h(\eta, y) = h(\eta_y + (\delta_{y_n})_y, y) - h(\eta_y, y).$$

By assumption on the order $<$ we cannot have $y_n < y$ so that $(\delta_{y_n})_y = 0$. Hence $D_{y_n} h(\eta, y) = 0$ and therefore also $h_n(y)(y_1, \dots, y_n) = 0$. By symmetry this remains true if $y < y_i$ for an arbitrary $i \in \{1, \dots, n\}$. The claim implies that

$$\mathbf{1}_{\Delta_{n+1}}(y_1, \dots, y_{n+1}) \tilde{h}_n(y_1, \dots, y_{n+1}) = \mathbf{1}_{\Delta_n}(y_1, \dots, y_n) \frac{1}{n+1} h_n(y_{n+1})(y_1, \dots, y_n),$$

where

$$\Delta_n := \{(y_1, \dots, y_n) \in \mathbb{Y}^n : y_1 < \dots < y_n\}. \tag{2.21}$$

In view of (2.1) it follows that

$$\begin{aligned}\|\tilde{h}_n\|_{n+1}^2 &= (n+1)!\|\mathbf{1}_{\Delta_{n+1}}\tilde{h}_n\|_{n+1}^2\\ &= \frac{(n+1)!}{(n+1)^2}\int \|\mathbf{1}_{\Delta_n}h_n(y)\|_n^2\lambda(\mathrm{d}y) = \frac{1}{n+1}\int \|h_n(y)\|_n^2\lambda(\mathrm{d}y).\end{aligned}$$

Hence we obtain from (2.14) and (2.15) that

$$\begin{aligned}\sum_{n=0}^{\infty}(n+1)!\|\tilde{h}_n\|_{n+1}^2 &= \sum_{n=0}^{\infty}\int n!\|h_n(y)\|_n^2\lambda(\mathrm{d}y)\\ &= \sum_{n=0}^{\infty}\int \mathbb{E}I_n(h_n(y))^2\lambda(\mathrm{d}y) = \int \mathbb{E}h(y)^2\lambda(\mathrm{d}y) < \infty.\end{aligned}$$

Therefore, (2.18) holds. □

Let $h \in L^2(\mathbb{P}_\eta \otimes \lambda)$ be predictable and $B \in \mathcal{Y}$. Then $\mathbf{1}_{\mathbf{N}\times B}h \in L^2(\mathbb{P}_\eta \otimes \lambda)$ is also predictable. Moreover, as noted above, we have from (2.20) that

$$\delta(\mathbf{1}_{\mathbf{N}\times B}h) = 0 \quad \mathbb{P}\text{-a.s. if } \lambda(B) = 0. \tag{2.22}$$

The following proposition implies a part of Theorem 2.1.

Proposition 2.5. *Let $h \in L^2(\mathbb{P}_\eta \otimes \lambda)$ be predictable. Then, for any $y \in \mathbb{Y}$,*

$$\mathbb{E}\left[\int h(\eta, z)\hat{\eta}(\mathrm{d}z)\,\middle|\,\eta_y\right] = \int_{y\downarrow} h(\eta, z)\hat{\eta}(\mathrm{d}z) \quad \mathbb{P}\text{-}a.s. \tag{2.23}$$

Proof. The right-hand side of (2.23) can be chosen $\sigma(\eta_y)$-measurable. This fact can be traced back to (2.14): if $f \in L^2(\lambda^n)$ is symmetric and vanishes outside B^n for some $B \in \mathcal{Y}$ and I_n^B denotes the nth Wiener–Itô integral with respect to the restriction of η to B, then $I_n(f) = I_n^B(f)$ $\mathbb{P}$-a.s.

To prove (2.23), we take $y \in \mathbb{Y}$ and a measurable function $g : \mathbf{N} \to \mathbb{R}$ such that the function g_y defined by $g_y(\mu) := g(\mu_y)$ satisfies (2.19). Since $D_z g_y = 0$ for $y < z$ we obtain from Proposition 2.3 that

$$0 = \mathbb{E}g(\eta_y)\int \mathbf{1}\{y < z\}h(\eta, z)\hat{\eta}(\mathrm{d}z).$$

From (2.22) and (2.1) we have

$$\int \mathbf{1}_{[y]}(z)h(\eta, z)\hat{\eta}(\mathrm{d}z) = 0 \quad \mathbb{P}\text{-a.s.} \tag{2.24}$$

Hence we obtain from the linearity of δ that

$$\mathbb{E}g(\eta_y)\int h(\eta, z)\hat{\eta}(\mathrm{d}z) = \mathbb{E}g(\eta_y)\int_{y\downarrow} h(\eta, z)\hat{\eta}(\mathrm{d}z). \tag{2.25}$$

Now we consider a function g of the form $g(\mu) := \exp[-\int h\mathrm{d}\mu]$, where $h : \mathbb{Y} \to \mathbb{R}_+$ is measurable and vanishes outside a set $C \in \mathcal{Y}$ with $\lambda(C) < \infty$. It can be easily checked, that g_y satisfies (2.19) (cf. also the proof of Theorem 3.3 in [20]). Hence (2.25) holds for all linear combinations of such functions. A monotone class argument shows that (2.25) holds for all bounded measurable $g : \mathbf{N} \to \mathbb{R}$ (cf. the proof of Lemma 2.2 in [20]). This is enough to deduce (2.23). □

Proof of Theorem 2.1. Let $f \in L^2(\mathbb{P}_\eta)$ and define $h : \mathbf{N} \times \mathbb{Y} \to \mathbb{R}$ by (2.4). Then h is predictable. Moreover, Theorem 1.5 in [20] implies that $h \in L^2(\mathbb{P}_\eta \otimes \lambda)$, that is (2.8) holds. By Proposition 2.4, the Kabanov–Skorohod integral $\delta(h)$ is well defined. We have to show that

$$f(\eta) = \mathbb{E}f(\eta) + \delta(h) \quad \mathbb{P}\text{-a.s.} \tag{2.26}$$

Let $g \in L^2(\mathbb{P}_\eta)$ satisfy (2.19). By Proposition 2.3,

$$\begin{aligned}\mathbb{E}g(\eta)\delta(h) &= \mathbb{E}\int D_y g(\eta)\mathbb{E}[D_y f(\eta)|\eta_y]\lambda(\mathrm{d}y)\\ &= \int \mathbb{E}[\mathbb{E}[D_y g(\eta)|\eta_y]\mathbb{E}[D_y f(\eta)|\eta_y]]\lambda(\mathrm{d}y),\end{aligned}$$

where the second equality comes from Fubini's theorem and a standard property of conditional expectations. Applying Theorem 1.5 in [20], we obtain that

$$\mathbb{E}g(\eta)\delta(h) = \mathbb{E}g(\eta)f(\eta) - (\mathbb{E}g(\eta))(\mathbb{E}f(\eta)),$$

that is $\mathbb{E}g(\eta)(\mathbb{E}f(\eta) + \delta(h)) = \mathbb{E}g(\eta)f(\eta)$. Since the set of all $g \in L^2(\mathbb{P}_\eta)$ satisfying (2.19) is dense in $L^2(\mathbb{P}_\eta)$, we obtain (2.26). The remaining assertion follows from Proposition 2.5. □

We have proved in [20] that

$$\delta(h) = \int h(\eta - \delta_y, y)\eta(\mathrm{d}y) - \int h(\eta, y)\lambda(\mathrm{d}y) \quad \mathbb{P}\text{-a.s.} \tag{2.27}$$

for any $h \in L^1(\mathbb{P}_\eta \otimes \lambda) \cap L^2(\mathbb{P}_\eta \otimes \lambda)$ satisfying (2.18). The main tool for the proof was the identity

$$\mathbb{E}\int g(\eta, y)\eta(\mathrm{d}y) = \mathbb{E}\int g(\eta + \delta_y, y)\lambda(\mathrm{d}y), \tag{2.28}$$

that holds for any $g \in L^1(\mathbb{P}_\eta \otimes \lambda)$; see Mecke [22]. If $h \in L^1(\mathbb{P}_\eta \otimes \lambda) \cap L^2(\mathbb{P}_\eta \otimes \lambda)$ is predictable, then Proposition 2.4 and (2.27) imply that (2.6) holds. (Note that $(\delta_y)_y = 0$ for all $y \in \mathbb{Y}$.) This is the first step for establishing the following alternative approach to the Kabanov–Skorohod integral of predictable functions. We denote the space of predictable elements of $L^2(\mathbb{P}_\eta \otimes \lambda)$ by $L^2_p(\mathbb{P}_\eta \otimes \lambda)$.

Theorem 2.6. *The restriction of δ to $L^2_p(\mathbb{P}_\eta \otimes \lambda)$ is the unique linear operator from $L^2_p(\mathbb{P}_\eta \otimes \lambda)$ to $L^2(\mathbb{P})$, satisfying (2.6) for predictable $h \in L^1(\mathbb{P}_\eta \otimes \lambda) \cap L^2(\mathbb{P}_\eta \otimes \lambda)$ and the isometry relation (2.7).*

Proof. We have already shown that (2.6) holds. To establish (2.7) for $h \in L^2_p(\mathbb{P}_\eta \otimes \lambda)$, we first assume that h is bounded and that $h(\mu, x) = 0$ for $x \notin C \in \mathcal{Y}$, where $\lambda(C) < \infty$. In particular, $h \in L^q(\mathbb{P}_\eta \otimes \lambda)$ for any $q > 0$. By (2.6),

$$\begin{aligned}\mathbb{E}\delta(h)^2 = \mathbb{E}&\left(\int h(\eta - \delta_y, y)\eta(\mathrm{d}y)\right)^2 - 2\mathbb{E}\left(\int h(\eta - \delta_x, x)\eta(\mathrm{d}x)\int h(\eta, y)\lambda(\mathrm{d}y)\right)\\ &+\mathbb{E}\left(\int h(\eta, y)\lambda(\mathrm{d}y)\right)^2.\end{aligned} \tag{2.29}$$

Our assumptions on h guarantee that all these expectations are finite. We now perform a fairly standard calculation based on the Mecke equation (2.28). The first term on the right-hand side of (2.29) equals

$$\begin{aligned}
&\mathbb{E}\int h(\eta,y)^2\eta(\mathrm{d}y)+\mathbb{E}\iint h(\eta,y)h(\eta,x)(\eta-\delta_x)(\mathrm{d}y)\eta(\mathrm{d}x)\\
&\quad=\mathbb{E}\int h(\eta,y)^2\lambda(\mathrm{d}y)+\mathbb{E}\iint h(\eta+\delta_y,x)h(\eta+\delta_x,y)\lambda(\mathrm{d}y)\lambda(\mathrm{d}x)\\
&\quad=\mathbb{E}\int h(\eta,y)^2\lambda(\mathrm{d}y)+2\mathbb{E}\iint \mathbf{1}\{x<y\}h(\eta,x)h(\eta+\delta_x,y)\lambda(\mathrm{d}y)\lambda(\mathrm{d}x),
\end{aligned}$$

where we have used (2.5) and $(\delta_y)_y=0$ in both equalities, as well as symmetry and (2.1) in the second equality. The second term on the right-hand side of (2.29) equals

$$\begin{aligned}
&-2\mathbb{E}\iint \mathbf{1}\{x<y\}h(\eta,x)h(\eta+\delta_x,y)\lambda(\mathrm{d}y)\lambda(\mathrm{d}x)\\
&\quad-2\mathbb{E}\iint \mathbf{1}\{y<x\}h(\eta,x)h(\eta,y)\lambda(\mathrm{d}y)\lambda(\mathrm{d}x).
\end{aligned}$$

Summarizing, we obtain that (2.7) holds, as required.

In the case of a general $h\in L^2_p(\mathbb{P}_\eta\otimes\lambda)$ we define, for $k\in\mathbb{N}$,

$$h_k(\mu,x):=\mathbf{1}\{|h(\mu,x)|\le k\}\mathbf{1}\{x\in C_k\}h(\mu,x),\quad (\mu,x)\in\mathbf{N}\times\mathbb{Y},$$

where $C_k\uparrow\mathbb{Y}$ and $\lambda(C_k)<\infty$. The functions h_k are predictable and satisfy the assumptions made above. From dominated convergence we have $\mathbb{E}\int(h(\eta,x)-h_k(\eta,x))^2\lambda(\mathrm{d}x)\to 0$ as $k\to\infty$. Then (2.7) implies that $\delta(h_k)$ is a Cauchy sequence in $L^2(\mathbb{P})$ and hence converges towards some $X\in L^2(\mathbb{P})$. Since δ is closed, we obtain $X=\delta(h)$ and hence the assertion.

Consider now another linear operator $\tilde\delta$ from $L^2_p(\mathbb{P}_\eta\otimes\lambda)$ to $L^2(\mathbb{P})$, satisfying (2.6) and (2.7). Take $h\in L^2_p(\mathbb{P}_\eta\otimes\lambda)$. As above we can choose a sequence of predictable functions $h_k\in L^1(\mathbb{P}_\eta\otimes\lambda)\cap L^2(\mathbb{P}_\eta\otimes\lambda)$, $k\in\mathbb{N}$, such that $h_k\to h$ in $L^2(\mathbb{P}_\eta\otimes\lambda)$. By linearity and (2.7),

$$\mathbb{E}(\tilde\delta(h)-\tilde\delta(h_k))^2=\mathbb{E}(\tilde\delta(h-h_k))^2=\mathbb{E}\int(h(\eta,y)-h_k(\eta,y))^2\lambda(\mathrm{d}y).$$

Therefore, $\tilde\delta(h_k)\to\tilde\delta(h)$ in $L^2(\mathbb{P})$. Similarly, $\delta(h_k)\to\delta(h)$ in $L^2(\mathbb{P})$. By (2.6) we have $\delta(h_k)=\tilde\delta(h_k)$ $\mathbb{P}$-a.s. for any $k\in\mathbb{N}$. This implies $\delta(h)=\tilde\delta(h)$ $\mathbb{P}$-a.s., as asserted in the theorem. □

By linearity and polarization we can extend (2.7) as follows.

Corollary 2.7. *Let $h,\tilde h\in L^2(\mathbb{P}_\eta\otimes\lambda)$ be predictable. Then*

$$\mathbb{E}\delta(h)\delta(\tilde h)=\mathbb{E}\int h(\eta,y)\tilde h(\eta,y)\lambda(\mathrm{d}y).\tag{2.30}$$

3. Martingales and stochastic integration

Assume that $\mathbb{Y}=\mathbb{R}_+\times\mathbb{X}$, where $(\mathbb{X},\mathcal{X})$ is a measurable space. We define $(s,x)<(s',x')$ if and only if $s<s'$. Throughout this section we consider a Poisson process η on $\mathbb{Y}$ whose intensity measure λ is σ-finite and satisfies (1.1). We discuss Theorem 2.1 and the Kabanov–Skorohod integral of predictable functions.

For any $s \geq 0$ and $\mu \in \mathbf{N}$ we denote by μ_s (resp. μ_{s-}) the restriction of μ to $[0, s] \times \mathbb{X}$ (resp. $[0, s) \times \mathbb{X}$). Theorem 2.1 takes the following form.

Theorem 3.1. *Let $f \in L^2(\mathbb{P}_\eta)$. Then*

$$\mathbb{E} \int \mathbb{E}[D_{(s,x)} f(\eta)|\eta_{s-}]^2 \lambda(\mathrm{d}(s, x)) < \infty \tag{3.1}$$

and we have for any $t \geq 0$ that $\mathbb{P}$-a.s.

$$\mathbb{E}[f(\eta)|\eta_t] = \mathbb{E} f(\eta) + \int \mathbf{1}_{[0,t]}(s) \mathbb{E}[D_{(s,x)} f(\eta)|\eta_{s-}] \hat{\eta}(\mathrm{d}(s, x)). \tag{3.2}$$

Proof. Relation (3.1) follows directly from Theorem 2.1. For any $t \geq 0$ we have $\mathbb{P}$-a.s. that

$$\mathbb{E}[f(\eta)|\eta_t] = \int f(\eta_t + \mu) \mathbb{P}_{\eta - \eta_t}(\mathrm{d}\mu),$$

where $\mathbb{P}_{\eta-\eta_t}$ is the distribution of the restriction of η to $(t, \infty) \times \mathbb{X}$. By (1.1), $\eta(\{t\} \times \mathbb{X}) = 0$ $\mathbb{P}$-a.s. Therefore, $\eta_t = \eta_{t-}$ a.s. and $\mathbb{P}_{\eta-\eta_t}$ is also the distribution of the restriction of η to $[t, \infty) \times \mathbb{X}$. Hence $\mathbb{E}[f(\eta)|\eta_t] = \mathbb{E}[f(\eta)|\eta_{t-}]$ and (3.2) follows from (2.10) and (2.24). $\square$

Remark 3.2. Let $h \in L^2(\mathbb{P}_\eta \otimes \lambda)$ be predictable and define

$$M_t := \int \mathbf{1}_{[0,t]}(s) h(\eta, s, x) \hat{\eta}(\mathrm{d}(s, x)), \quad t \in [0, \infty].$$

Proposition 2.5 and (2.24) imply for any $t \in [0, \infty]$ that $\mathbb{E}[M_\infty|\eta_{t-}] = M_t$ $\mathbb{P}$-a.s. In the proof of Theorem 3.1 we have seen that $\mathbb{E}[M_\infty|\eta_{t-}] = \mathbb{E}[M_\infty|\eta_t]$ $\mathbb{P}$-a.s. Hence $(M_t)_{t\in[0,\infty]}$ is a martingale with respect to the filtration $(\sigma(\eta_t))_{t\in[0,\infty]}$, where $\eta_\infty := \eta$. This martingale is *square integrable*, that is $M_\infty \in L^2(\mathbb{P})$.

Our next aim is to clarify the meaning of the predictability property (2.5) and to discuss the Kabanov–Skorohod integral of predictable functions. To do so, we introduce a measurable subset $\mathbf{N}^*$ of $\mathbf{N}$ as follows. Let $C_1, C_2, \ldots$ be a sequence of disjoint measurable subsets of $\mathbb{Y}$ with union $\mathbb{Y}$. Let $\mathbf{N}^*$ be the set of all $\mu \in \mathbf{N}$ having the properties $\mu(\{0\} \times \mathbb{X}) = 0$ and $\mu(C_n) < \infty$ for all $n \in \mathbb{N}$. For any $t \in [0, \infty]$ let $\mathcal{N}_t$ the smallest σ-field of subsets of $\mathbf{N}^*$, making the mappings $\mu \mapsto \mu(B \cap ([0, t] \times \mathbb{X}))$ measurable for all $B \in \mathcal{Y}$. Here $\mu_\infty := \mu$. The *predictable σ-field $\mathcal{P}$* (see [13]) is the smallest σ-field containing the sets

$$A \times (s, t] \times B, \quad s < t, A \in \mathcal{N}_s, B \in \mathcal{X}. \tag{3.3}$$

The next proposition provides a useful characterization of the predictable σ-field. We have to assume that $(\mathbb{X}, \mathcal{X})$ is Borel isomorphic to a Borel subset of $[0, 1]$. Such a space is called *Borel space*; see [16].

Proposition 3.3. *Assume that $(\mathbb{X}, \mathcal{X})$ is a Borel space. Let $h : \mathbf{N}^* \times \mathbb{R}_+ \times \mathbb{X} \to \mathbb{R}$ be measurable. Then h is $\mathcal{P}$-measurable if and only if (2.5) holds, that is*

$$h(\mu, s, x) = h(\mu_{s-}, s, x), \quad (\mu, s, x) \in \mathbf{N}^* \times \mathbb{X} \times \mathbb{R}_+. \tag{3.4}$$

Proof. The filtration $(\mathcal{N}_t)_{t\geq 0}$ is not right-continuous, but has otherwise many of the properties of a point process filtration as studied in Section 2.2 of [19]. To make this more precise, we introduce $\mathbf{N}_n$, $n \in \mathbb{N}$, as the set of all finite integer-valued measures μ on C_n such that $\mu((\{0\} \times \mathbb{X}) \cap C_n) = 0$. Any $\mu \in \mathbf{N}_n$ can be written as

$$\mu(B) = \sum_{i=1}^{m} \int \mathbf{1}_B(s_i, x)\mu_i(\mathrm{d}x), \tag{3.5}$$

where $m \geq 0$, $0 < s_1 < \cdots < s_m$, and $\mu_1, \ldots, \mu_m$, are finite non-trivial integer-valued measures on $\mathbb{X}$. (Here we use the Borel structure of $\mathbb{X}$.) It is convenient to identify μ with the infinite sequence (s_i, μ_i), $i \in \mathbb{N}$, where $(s_i, \mu_i) := (\infty, 0)$ for $i > m$, and 0 denotes the zero measure. Let $\mathbf{N}_f(\mathbb{X})$ be the space of all finite counting measures on $\mathbb{X}$. It can be proved that this is a Borel space. Moreover, the quantities $m, s_1, \ldots, s_m, \mu_1, \ldots, \mu_m$ in (3.5) depend on μ in a measurable way. (This does require the Borel structure of $\mathbb{X}$ and $\mathbf{N}_f(\mathbb{X})$.) Therefore, we can identify $\mathbf{N}_n$ with a measurable subset $\mathbf{N}'_n$ of the space $\mathbf{M}$ defined as the set of all sequences $((s_i, \mu_i))_{i\in\mathbb{N}} \in ((0, \infty] \times \mathbf{N}_f(\mathbb{X}))^\infty$ with the following properties. If $s_i < \infty$, then $s_i < s_{i+1}$ and $\mu_i \neq 0$. If $s_i = \infty$, then $s_{i+1} = \infty$ and $\mu_i = 0$. The space $\mathbf{N}'_n \subset \mathbf{M}$ can be equipped with the product topology inherited from $([0, \infty] \times \mathbf{N}_f(\mathbb{X}))^\infty$. Now we identify the whole space $\mathbf{N}^*$ with $\mathbf{N}'_1 \times \mathbf{N}'_2 \times \cdots$, again equipped with the product topology. The crucial property of this topology is that the mappings $s \mapsto \mu_s$ and $s \mapsto \mu_{s-}$ are right-continuous respectively left-continuous. Therefore, it is not difficult to check that Theorem 2.2.6 in [19] applies to the filtration $(\mathcal{N}_t)$. □

Remark 3.4. The assumption $\mu(\{0\} \times \mathbb{X}) = 0$ for $\mu \in \mathbf{N}^*$ has been made for convenience. Without this condition the σ-field $\mathcal{N}_0$ becomes non-trivial, and we have to include the sets $A \times \{0\} \times B$ ($A \in \mathcal{N}_0$, $B \in \mathcal{X}$) into the σ-field $\mathcal{P}$. If we then redefine μ_{0-} as the restriction of μ to $\{0\} \times \mathbb{X}$, Proposition 3.3 remains valid.

We now assume that the sets C_n, $n \in \mathbb{N}$, are chosen in such a way, that the intensity measure λ of η is finite on these sets. Let η^* be the random element in $\mathbf{N}^*$, defined by $\eta^* := \eta$ if $\eta \in N^*$ and $\eta^* := 0$, otherwise. The second case has probability 0. Let F_1^* and F_2^* denote the $\mathcal{P}$-measurable elements of $L^1(\mathbb{P}_{\eta^*} \otimes \lambda)$ and $L^2(\mathbb{P}_{\eta^*} \otimes \lambda)$ respectively. For $h \in F_2^*$ we can define the stochastic integral $\delta^*(h)$ of h against the compensated Poisson process $\eta^* - \lambda$ in the following standard way; see e.g. [9]. If $h \in F_1^* \cap F_2^*$ we define

$$\delta^*(h) := \int h(\eta^*, s, x)\eta^*(\mathrm{d}(s, x)) - \int h(\eta^*, s, x)\lambda(\mathrm{d}(s, x)). \tag{3.6}$$

In particular,

$$\delta^*(\mathbf{1}_{A\times(s,t]\times B}\mathbf{1}_{\mathbf{N}^*\times C_n}) = \mathbf{1}_A(\eta^*)(\eta^*(((s, t] \times B) \cap C_n) - \lambda(((s, t] \times B) \cap C_n)), \tag{3.7}$$

where $s < t$, $A \in \mathcal{N}_s$, $n \in \mathbb{N}$, and $B \in \mathcal{X}$. Let $h \in F_1^* \cap F_2^*$ and define $\tilde{h} : \mathbf{N} \times \mathbb{Y} \to \mathbb{R}$ by $\tilde{h} := h$ on $\mathbf{N}^* \times \mathbb{Y}$ and $\tilde{h} := 0$, otherwise. By Proposition 3.3, $\tilde{h}$ is predictable. Since $\mathbb{P}(\eta \in \mathbf{N}^*) = 1$ we obtain from (2.6) that $\delta^*(h) = \delta(\tilde{h})$ $\mathbb{P}$-a.s. Therefore, (2.7) implies the isometry relation

$$\mathbb{E}\delta^*(h)^2 = \mathbb{E} \int h(\eta^*, s, x)^2\lambda(\mathrm{d}(s, x)) \tag{3.8}$$

for any $h \in F_1^* \cap F_2^*$. Since $F_1^* \cap F_2^*$ is dense in F_2^* we can extend δ^* to a linear operator from F_2^* to $L^2(\mathbb{P})$. Eq. (3.8) remains valid for arbitrary $h \in F_2^*$.

We now prove that δ extends the stochastic integral δ^*. Special cases of this result can be found in [14,23]; see also [26, Proposition 3.7] for the case of Lévy processes. For $h : \mathbf{N} \times \mathbb{Y} \to \mathbb{R}$, the function $h^* : \mathbf{N}^* \times \mathbb{Y} \to \mathbb{R}$ denotes the restriction of h to $\mathbf{N}^* \times \mathbb{Y}$.

Theorem 3.5. *Let $h \in L^2(\mathbb{P}_\eta \otimes \lambda)$ such that h^* is $\mathcal{P}$-measurable. Then $\delta(h) = \delta^*(h^*)$ $\mathbb{P}$-a.s.*

Proof. Since $\mathbb{P}(\eta \in \mathbf{N}^*) = 1$, we have from Proposition 2.4 that $\delta(h)$ is defined. By (2.6) and (3.6) (and Proposition 3.3) the assertion holds for any $h \in F_1^* \cap F_2^*$. In the general case we may choose $h_k \in F_1^* \cap F_2^*$, $k \in \mathbb{N}$, such that $h_k \to h$ as $k \to \infty$ in $L^2(\mathbb{P}_\eta \otimes \lambda)$. Then $\delta^*(h_k^*) = \delta(h_k)$ converges to $\delta^*(h^*)$ in $L^2(\mathbb{P})$. Since δ is closed, this yields the assertion. $\square$

Remark 3.6. Proposition 3.3 justifies our terminology for measurable functions h on $\mathbf{N} \times \mathbb{Y}$ satisfying (2.5). By this proposition, if h is predictable then h^* is $\mathcal{P}$-measurable. Conversely, if h^* is $\mathcal{P}$-measurable then there exists predictable $\tilde{h}$ with $\tilde{h}^* = h^*$. If $h \in L^2(\mathbb{P}_\eta \otimes \lambda)$ is predictable then our notation $\int h d\hat{\eta} := \delta(h)$ is justified by Theorem 3.5.

Remark 3.7. A standard assumption in the stochastic analysis literature is completeness of the underlying filtration. Quite often one can find no further comment on this technical (and sometimes annoying) hypothesis. In this paper we do not make this completeness assumption, which is rather alien to point process theory.

4. Independent random measures

Let $(\mathbb{Y}', \mathcal{Y}')$ be a Borel space and β be a σ-finite measure and diffuse measure on $\mathbb{Y}'$. Let $\mathcal{Y}'_0$ denote the system of all sets $B \in \mathcal{Y}'$ such that $\beta(B) < \infty$. In this section we consider an *independent random measure* on $\mathbb{Y}'$ (see [11]) with *variance measure* β. This is a family $\zeta' := \{\zeta'(B) : B \in \mathcal{Y}'_0\}$ with the following three properties. First, $\mathbb{E}\zeta'(B) = 0$ and $\mathbb{E}\zeta'(B)^2 = \beta(B)$ for any $B \in \mathcal{Y}'_0$. Second, if $B_1, B_2, \ldots \in \mathcal{Y}'_0$ are pairwise disjoint, then $\zeta'(B_1), \zeta'(B_2), \ldots$ are independent. Third, if $B_1, B_2, \ldots \in \mathcal{Y}'_0$ are pairwise disjoint and $B := \cup B_n \in \mathcal{Y}'_0$ then $\zeta'(B) = \sum_n \zeta'(B_n)$ in $L^2(\mathbb{P})$. By [16, Theorem 4.1] the series also converges almost surely. Since β is diffuse, it follows that the distribution of $\zeta'(B)$ is infinitely divisible for any $B \in \mathcal{Y}'_0$; see [17, p. 81] for a closely related argument. The Lévy–Khinchin representation (see [16, Corollary 15.8]) implies that

$$\log \mathbb{E}e^{iu\zeta'(B)} = -a_B u^2 + \int (e^{iuz} - 1 - iuz)\lambda(B, dz), \quad u \in \mathbb{R}, \tag{4.1}$$

where $a_B \in \mathbb{R}$ and $\lambda(B, \cdot)$ is a measure on $\mathbb{R}^* := \mathbb{R} \setminus \{0\}$ satisfying $\int z^2 \lambda(B, dz) = \beta(B)$. The measure $\lambda(B, \cdot)$ is the *Lévy measure* of $\zeta'(B)$ and is unique. We assume that $a_B = 0$, so that ζ has no Gaussian component. If $B \in \mathcal{Y}'_0$ is the disjoint union of measurable sets B_n, $n \in \mathbb{N}$, then the independence of the $\zeta'(B_n)$ and the uniqueness of the Lévy measure implies that $\lambda(B, \cdot) = \sum_{n=1}^{\infty} \lambda(B_n, \cdot)$. By a well-known result from measure theory (see [17, p. 82]) there is a unique measure λ on $\mathbb{Y}' \times \mathbb{R}^*$ such that $\lambda(B \times C) = \lambda(B, C)$ for all $B \in \mathcal{Y}'_0$ and all measurable $C \subset \mathbb{R}^*$. Hence Eq. (4.1) can be rewritten as

$$\log \mathbb{E}e^{iu\zeta'(B)} = \int \mathbf{1}_B(x)(e^{iuz} - 1 - iuz)\lambda(d(x, z)), \quad u \in \mathbb{R}, \tag{4.2}$$

whenever $\beta(B) < \infty$. By definition,

$$\int z^2 \mathbf{1}\{x \in \cdot\}\lambda(d(x, z)) = \beta(\cdot). \tag{4.3}$$

In particular, λ is σ-finite.

Let us now consider a Poisson process η on $\mathbb{Y} := \mathbb{Y}' \times \mathbb{R}^*$ with intensity measure λ. For any $B \in \mathcal{Y}'_0$ we define the Wiener–Itô integral

$$\zeta(B) := \int z\mathbf{1}_B(y)\hat{\eta}(\mathrm{d}(y,z)). \tag{4.4}$$

Then $\zeta := \{\zeta(B) : B \in \mathcal{Y}'_0\}$ is an independent random measure with variance measure β. We might think of a point of η as being a point in $\mathbb{Y}'$ with the second coordinate representing its weight. Then the integral (4.4) is the weighted sum of all points lying in B, suitably compensated. It follows from (4.2) and basic properties of η (cf. [16, Lemma 12.2] or [17, Section 3.2]) that $\zeta(B)$ and $\zeta'(B)$ have the same distribution for any $B \in \mathcal{Y}'_0$. Henceforth it is convenient to work with ζ and the Poisson process η.

We now assume that $<'$ is a partial ordering on $\mathbb{Y}'$ satisfying the assumptions listed in Section 2, where in (2.1) the measure λ has to be replaced with β. (Since $y \in [y]$ for all $y \in \mathbb{Y}'$ this is strengthening the diffuseness assumption on β.) Then we can define a binary relation $<$ on $\mathbb{Y} = \mathbb{Y}' \times \mathbb{R}^*$ by setting $(y,z) < (y',z')$ if $y <' y'$. This relation also satisfies our assumptions, where (2.1) comes from (4.3) and the assumption on β. The measurability of $(\mu, y) \mapsto \mu_y$ can be proved using a measurable disintegration $\mu(\mathrm{d}(y,z)) = K(\mu, y, \mathrm{d}z)\mu^*(\mathrm{d}y)$, where K is a kernel from $\mathbf{N} \times \mathbb{Y}'$ to $\mathbb{R}^*$ and $\mu \mapsto \mu^*$ is a measurable mapping from $\mathbf{N} = \mathbf{N}(\mathbb{Y})$ to $\mathbf{N}(\mathbb{Y}')$ such that $\mu(\cdot \times \mathbb{R}^*)$ and μ^* are equivalent measures for all $\mu \in \mathbf{N}$.

The stochastic integral of a predictable function $h : \mathbf{N} \times \mathbb{Y}' \to \mathbb{R}$ against ζ is defined by

$$\int h(\eta, y)\zeta(\mathrm{d}y) := \int zh(\eta, y)\hat{\eta}(\mathrm{d}(y,z)) \tag{4.5}$$

provided that

$$\mathbb{E}\int h(\eta, y)^2\beta(\mathrm{d}y) = \mathbb{E}\int z^2 h(\eta, y)^2\lambda(\mathrm{d}(y,z)) < \infty. \tag{4.6}$$

Let $\mathcal{M}^2_\zeta \subset L^2(\mathbb{P})$ be the space of all square integrable random variables X given by

$$X = \int h(\eta, y)\zeta(\mathrm{d}y), \tag{4.7}$$

where the predictable function h satisfies (4.6). The isometry relation (2.7) and the completeness of $L^2(\mathbb{P}_\eta \otimes \beta)$ imply that $\mathcal{M}^2_\zeta$ is a closed linear space. Hence any $Y \in L^2(\mathbb{P})$ can be uniquely written as $Y = X + X'$, where $X \in \mathcal{M}^2_\zeta$ and $X' \in L^2(\mathbb{P})$ is orthogonal to $\mathcal{M}^2_\zeta$. Decompositions of this type were first considered by Kunita and Watanabe [18]. The following theorem makes this decomposition more explicit. We use a stochastic kernel $J(y, \mathrm{d}z)$ from $\mathbb{Y}'$ to $\mathbb{R}^*$ such that

$$z^2\lambda(\mathrm{d}(y,z)) = J(y, \mathrm{d}z)\beta(\mathrm{d}y). \tag{4.8}$$

Such a kernel exists by a standard disintegration result (cf. [16, Theorem 6.3] for a special case).

Theorem 4.1. *Let $f \in L^2(\mathbb{P}_\eta)$ and define a predictable $h_f : \mathbf{N} \times \mathbb{Y}' \to \mathbb{R}$ by*

$$h_f(\eta, y) = \mathbb{E}\left[\int z^{-1} D_{(y,z)} f(\eta) J(y, \mathrm{d}z) \,\middle|\, \eta_y\right]. \tag{4.9}$$

Then h_f satisfies (4.6) and we have $\mathbb{P}$-a.s. that

$$f(\eta) = \mathbb{E} f(\eta) + \int h_f(\eta, y)\zeta(\mathrm{d}y) + X', \tag{4.10}$$

where $X' \in L^2(\mathbb{P})$ is orthogonal to $\mathcal{M}^2_\zeta$.

Proof. By Fubini's theorem applied to kernels we have

$$\mathbb{E} \int h_f(\eta, y)^2 \beta(\mathrm{d}y) = \int \mathbb{E} \left(\int \mathbb{E}[z^{-1} D_{(y,z)} f(\eta)|\eta_y] J(y, \mathrm{d}z) \right)^2 \beta(\mathrm{d}y).$$

Applying Jensen's inequality to the stochastic kernel $J(y, \mathrm{d}z)$ and using (2.8) and (4.8) gives (4.6). We now define $X' \in L^2(\mathbb{P})$ by

$$X' := \int (\mathbb{E}[D_{(y,z)} f(\eta)|\eta_y] - z h_f(\eta, y))\hat{\eta}(\mathrm{d}(y, z)). \tag{4.11}$$

Theorem 3.1 implies (4.10). It remains to show that X' is orthogonal to $\mathcal{M}^2_\zeta$. To this end, we consider a random variable X as given in (4.7). By Corollary 2.7,

$$\mathbb{E} X X' = \mathbb{E} \int z h(y)(\mathbb{E}[D_{(y,z)} f(\eta)|\eta_y] - z h_f(\eta, y))\lambda(\mathrm{d}(y, z)). \tag{4.12}$$

We have

$$\begin{aligned}
\mathbb{E} \int z^2 h(y) h_f(\eta, y)\lambda(\mathrm{d}(y, z)) &= \mathbb{E} \int h(y) h_f(\eta, y)\beta(\mathrm{d}y) \\
&= \mathbb{E} \iint h(y)\mathbb{E}[z^{-1} D_{(y,z)} f(\eta)|\eta_y] J(s, x, \mathrm{d}z)\beta(\mathrm{d}(s, x)) \\
&= \mathbb{E} \int z h(y)\mathbb{E}[D_{(y,z)} f(\eta)|\eta_y]\lambda(\mathrm{d}(y, z)).
\end{aligned}$$

Hence (4.12) implies $\mathbb{E} X X' = 0$, as claimed. □

Di Nunno [5] proved Theorem 4.1 for special ("core") functions f (and allowing also for a Gaussian part of ζ) in case $\mathbb{Y}' = \mathbb{R}_+ \times \mathbb{X}$, with $<'$ given as in Section 3. In the case where $J(y, \cdot) = \delta_1$ for β-a.e. y (that is that ζ has only atoms of size 1), (4.10) reduces to the Clark–Ocone type formula (2.9).

The following result characterizes the class of square integrable stochastic integrals against ζ.

Corollary 4.2. *Let $f \in L^2(\mathbb{P}_\eta)$ such that $\mathbb{E} f(\eta) = 0$. Then $f(\eta) \in \mathcal{M}^2_\zeta$ if and only if there is some predictable $h : \mathbf{N} \times \mathbb{Y}' \to \mathbb{R}$ satisfying (4.6) such that*

$$\mathbb{E}[D_{(y,z)} f(\eta)|\eta_y] = z h(\eta, y) \quad \lambda\text{-a.e. } (y, z),\ \mathbb{P}\text{-a.s.} \tag{4.13}$$

Proof. Assume that (4.13) holds. Then $h = h_f$ and the random variable X' defined by (4.11) vanishes almost surely. Therefore, Theorem 4.1 shows that $f(\eta)$ can be written as a stochastic integral against ζ.

Assume conversely that $f(\eta) \in \mathcal{M}^2_\zeta$ and consider the decomposition (4.10). Since the orthogonal projection onto $\mathcal{M}^2_\zeta$ is unique, it follows that $X' = 0$ $\mathbb{P}$-a.s. By definition (4.11) this means that (4.13) holds with $h := h_f$. □

5. Minimal variance hedging

We consider a Poisson process η on $\mathbb{Y} := \mathbb{R}_+ \times \mathbb{X} \times \mathbb{X}'$, where $(\mathbb{X}, \mathcal{X})$ and $(\mathbb{X}', \mathcal{X}')$ are Borel spaces. The partial ordering on $\mathbb{Y}$ is defined by $(s, x, z) < (s', x', z')$ if $s < s'$. As always, the intensity measure λ of η is assumed to satisfy (2.1). Our aim in this section is to extend the results of Section 4 for the case $\mathbb{Y}' = \mathbb{R}_+ \times \mathbb{X}$. We replace $\mathbb{R}^*$ by the general space $\mathbb{X}'$ and the independent random measure ζ by a more general L^2-valued signed random measure. The special structure of $\mathbb{Y}'$ (and $\mathbb{Y}$) allows for a financial interpretation of our results. We consider a point (s, x, z) of η as representing a financial event at time s of (asset) type x and with mark z. Note that we allow for a continuum of assets. Given (s, x) and the past η_{s-}, the mark z might determine the price increment of asset x at time s. More formally, we let $\kappa : \mathbf{N} \times \mathbb{Y} \to \mathbb{R}$ be a predictable function and interpret $\kappa(\eta, s, x, z)$ as the size of the event (s, x, z). We assume that

$$\bar{\beta}(\cdot) := \mathbb{E} \int \kappa(\eta, s, x, z)^2 \mathbf{1}\{(s, x) \in \cdot\} \lambda(\mathrm{d}(s, x, z)) \tag{5.1}$$

is a σ-finite measure. The system of all measurable $B \subset \mathbb{R}_+ \times \mathbb{X}$ such that $\bar{\beta}(B) < \infty$ is denoted by $\mathcal{Y}'_0$. For any $B \in \mathcal{Y}'_0$ we define by

$$\zeta(B) := \int \kappa(\eta, s, x, z) \mathbf{1}_B(s, x) \hat{\eta}(\mathrm{d}(s, x, z)) \tag{5.2}$$

a square integrable random variable having $\mathbb{E}\zeta(B) = 0$. The stochastic integral of a predictable $h : \mathbf{N} \times \mathbb{R}_+ \times \mathbb{X} \to \mathbb{R}$ (here $(s, x) < (s', x')$ if $s < s'$) against ζ is defined by

$$\int h(\eta, s, x) \zeta(\mathrm{d}(s, x)) := \int h(\eta, s, x) \kappa(\eta, s, x, z) \hat{\eta}(\mathrm{d}(s, x, z)) \tag{5.3}$$

provided that

$$\mathbb{E} \int h(\eta, s, x)^2 \kappa(\eta, s, x, z)^2 \lambda(\mathrm{d}(s, x, z)) < \infty. \tag{5.4}$$

We denote by $\mathcal{A}$ the set of all such predictable functions h.

Remark 5.1. Let $\mathcal{X}_0$ denote the system of all $B \in \mathcal{X}$ such that $[0, t] \times B \in \mathcal{Y}'_0$ for all $t \geq 0$. For $B \in \mathcal{X}_0$ we can define the square integrable martingale (see Remark 3.2)

$$\zeta_t(B) := \int \kappa(\eta, s, x, z) \mathbf{1}_{[0,t]}(s) \mathbf{1}_B(x) \hat{\eta}(\mathrm{d}(s, x, z)), \quad t \in [0, \infty].$$

We interpret $\zeta_t(B)$ as the (discounted) price of the assets in B at time t. Note that $\zeta_t(\cdot)$ is a signed measure on $\mathcal{X}_0$ in an L^2-sense. An element $h \in \mathcal{A}$ can be interpreted as admissible portfolio investing the amount $h(\eta, s, x)$ in asset x at time s. Accordingly, if the bond price is constant, and $V_0 \in \mathbb{R}$ then

$$V_t := V_0 + \int \mathbf{1}_{[0,t]}(s) h(\eta, s, x) \zeta(\mathrm{d}(s, x)), \quad t \in [0, \infty],$$

is the value process of the self-financing portfolio associated with h and an initial value V_0.

Let $f \in L^2(\mathbb{P}_\eta)$. We interpret $f(\eta)$ as a *claim* to be hedged (or approximated) by a random variable of the form $\mathbb{E}f(\eta) + \int h(\eta, s, x)\zeta(\mathrm{d}(s, x))$ with $h \in \mathcal{A}$. A *minimal variance hedge* of

$f(\eta)$ is then a portfolio $h_f \in \mathcal{A}$ satisfying

$$\mathbb{E}\left(f(\eta) - \mathbb{E}f(\eta) - \int h_f(\eta, s, x)\zeta(\mathrm{d}(s, x))\right)^2$$
$$= \inf_{h \in \mathcal{A}} \mathbb{E}\left(f(\eta) - \mathbb{E}f(\eta) - \int h(\eta, s, x)\zeta(\mathrm{d}(s, x))\right)^2. \tag{5.5}$$

Remark 5.2. Problem (5.5) requires us to minimize the quadratic risk among all self-financing portfolios with initial value $\mathbb{E}f(\eta)$. We might also be interested in minimizing

$$\mathbb{E}\Big(f(\eta) - c - \int h(\eta, s, x)\zeta(\mathrm{d}(s, x))\Big)^2 \tag{5.6}$$

in $c \in \mathbb{R}$ and $h \in \mathcal{A}$. However, if $h_f \in \mathcal{A}$ solves (5.5) then the pair $(\mathbb{E}f(\eta), h_f)$ minimizes (5.6).

To solve (5.5) we need to generalize the disintegration (4.8). A kernel J from $\mathbf{N} \times \mathbb{R}_+ \times \mathbb{X}$ to $\mathbb{X}'$ is called predictable, if $(\mu, s, x) \mapsto J(\mu, s, x, C)$ is predictable for all $C \in \mathcal{X}'$. In the next proof and also later we use the generalized inverse $a^\oplus$ of a real number a. It is defined by $a^\oplus := a^{-1}$ if $a \neq 0$ and $a^\oplus := 0$ if $a = 0$.

Lemma 5.3. *There exists a predictable stochastic kernel J from $\mathbf{N} \times \mathbb{R}_+ \times \mathbb{X}$ to $\mathbb{X}'$ such that*

$$\kappa(\eta, s, x, z)^2\lambda(\mathrm{d}(s, x, z)) = J(\eta, s, x, \mathrm{d}z)\beta(\mathrm{d}(s, x)) \quad \mathbb{P}\text{-}a.s., \tag{5.7}$$

where the random measure β on $\mathbb{R}_+ \times \mathbb{X}$ is defined by

$$\beta(\cdot) := \int \mathbf{1}\{(s, x) \in \cdot\}\kappa(\eta, s, x, z)^2\lambda(\mathrm{d}(s, x, z)). \tag{5.8}$$

Proof. Define a measure $\bar{\lambda}$ on $\mathbb{Y}$ by

$$\bar{\lambda}(\mathrm{d}(s, x, z)) := \bar{\kappa}(s, x, z)\lambda(\mathrm{d}(s, x, z)),$$

where $\bar{\kappa}(s, x, z) := \mathbb{E}\kappa(\eta, s, x, z)^2$. Because the measure $\bar{\beta} = \bar{\lambda}(\cdot \times \mathbb{X}')$ (see (5.1)) is assumed to be σ-finite and $\mathbb{X}'$ is Borel, there is a stochastic kernel $\bar{J}$ from $\mathbb{R}_+ \times \mathbb{X}$ to $\mathbb{X}'$ such that

$$\bar{\lambda}(\mathrm{d}(s, x, z)) = \bar{J}(s, x, \mathrm{d}z)\bar{\beta}(\mathrm{d}(s, x)).$$

It follows that

$$\kappa(\eta, s, x, z)^2\lambda(\mathrm{d}(s, x, z)) = \bar{\kappa}(s, x, z)^\oplus\kappa(\eta, s, x, z)^2\bar{J}(s, x, \mathrm{d}z)\bar{\beta}(\mathrm{d}(s, x)) \quad \mathbb{P}\text{-a.s.} \tag{5.9}$$

In particular, the random measure β defined by (5.8) coincides a.s. with $g(\eta, s, x)\bar{\beta}(\mathrm{d}(s, x))$, where

$$g(\mu, s, x) := \int \bar{\kappa}(s, x, z)^\oplus\kappa(\mu, s, x, z)^2\bar{J}(s, x, \mathrm{d}z).$$

We now define

$$J(\mu, s, x, \mathrm{d}z) := g(\mu, s, x)^{-1}\bar{\kappa}(s, x, z)^\oplus\kappa(\mu, s, x, z)^2\bar{J}(s, x, \mathrm{d}z),$$

if $g(\mu, s, x) > 0$. Otherwise, let $J(\mu, s, x, \cdot)$ equal some fixed probability measure on $\mathbb{X}'$. Then J is predictable and (5.9) implies (5.7). $\square$

As in Section 4, let $\mathcal{M}^2_\zeta$ denote the space of all square integrable random variables that can be written as a stochastic integral (5.3).

Theorem 5.4. *Let $f \in L^2(\mathbb{P}_\eta)$ and define*

$$h_f(\eta, s, x) = \int \kappa(\eta, s, x, z)^{\oplus}\mathbb{E}[D_{(s,x,z)} f(\eta)|\eta_{s-}]J(\eta, s, x, \mathrm{d}z), \tag{5.10}$$

where the stochastic kernel J is as in Lemma 5.3. Then $h_f \in \mathcal{A}$ and (5.5) holds. Moreover, we have for any $t \in [0, \infty]$ that $\mathbb{P}$-a.s.

$$\mathbb{E}[f(\eta)|\eta_t] = \mathbb{E}f(\eta) + \int \mathbf{1}_{[0,t]}(s)h_f(\eta, s, x)\zeta(\mathrm{d}(s, x)) + N_t, \tag{5.11}$$

where (N_t) is a square integrable martingale such that N_∞ is orthogonal to $\mathcal{M}^2_\zeta$.

Proof. Clearly h_f is predictable. The integrability condition (5.4) can be checked exactly as in the proof of Theorem 4.1. We can now proceed as in the proof of Theorem 4.1 to derive the representation

$$f(\eta) = \mathbb{E}f(\eta) + \int h_f(\eta, s, x)\zeta(\mathrm{d}(s, x)) + X', \tag{5.12}$$

where $X' \in L^2(\mathbb{P})$ is orthogonal to $\mathcal{M}^2_\zeta$. This orthogonality implies (5.5). Let $t \geq 0$ and define $N_t := \mathbb{E}[X'|\eta_t]$. Taking conditional expectations in (5.12) and using Remark 3.2 yield (5.11). □

The next result characterizes the claims that can be perfectly hedged. The proof is an obvious generalization of the proof of Corollary 4.2.

Theorem 5.5. *Let $f \in L^2(\mathbb{P}_\eta)$. Then (5.5) vanishes if and only if there is some $h \in \mathcal{A}$ such that*

$$\mathbb{E}[D_{(s,x,z)} f(\eta)|\eta_{s-}] = \kappa(\eta, s, x, z)h(\eta, s, x) \quad \lambda\text{-a.e. } (s, x, z),\ \mathbb{P}\text{-a.s.} \tag{5.13}$$

In this case we have $h(\eta, s, x) = h_f(\eta, s, x)$ for $\bar{\beta}$-a.e. (s, x) and $\mathbb{P}$-a.s.

In the remainder of this section we assume that $\mathbb{X} = \mathbb{N}$, that is, we assume that there are only countably many assets. For any $j \in \mathbb{N}$ we define a measure λ_j on $\mathbb{R}_+ \times \mathbb{X}'$ by

$$\lambda_j := \iint \mathbf{1}\{(s, z) \in \cdot\}\lambda(\mathrm{d}s \times \{j\} \times \mathrm{d}z).$$

Because λ is σ-finite all measures λ_j must be σ-finite as well. Hence there exist σ-finite kernels J_j from $\mathbb{R}_+$ to $\mathbb{X}'$ and σ-finite measures μ_j on $\mathbb{R}_+$ satisfying

$$\lambda_j(\mathrm{d}(s, z)) = J_j(s, \mathrm{d}z)\mu_j(\mathrm{d}s), \quad j \in \mathbb{N}.$$

The predictable function κ is assumed to satisfy

$$\mathbb{E}\int \kappa(\eta, s, j, z)^2\lambda_j(\mathrm{d}(s, z)) < \infty, \quad j \in \mathbb{N}.$$

This implies the σ-finiteness of the measure (5.1). The kernel J of Lemma 5.3 is given by

$$J(\mu, s, j, \mathrm{d}z) = \left(\int \kappa(\mu, s, j, z)^2 J_j(s, \mathrm{d}z)\right)^{-1} \kappa(\mu, s, j, z)^2 J_j(s, \mathrm{d}z)$$

whenever $\int \kappa(\mu, s, j, z)^2 J_j(s, \mathrm{d}z) > 0$. If $f \in L^2(\mathbb{P}_\eta)$ then, according to Theorem 5.4, the minimal variance hedge h_f of $f(\eta)$ can be computed as

$$h_f(\eta, s, j) = \left(\int \kappa(\eta, s, j, z)^2 J_j(s, \mathrm{d}z)\right)^{\oplus} \times \int \kappa(\eta, s, j, z)\mathbb{E}[D_{(s,j,z)} f(\eta)|\eta_{s-}] J_j(s, \mathrm{d}z). \tag{5.14}$$

Example 5.6. Assume that $\mathbb{X}' = \mathbb{R}^*$ and that

$$\int z^2 \lambda_j([0, t] \times \mathrm{d}z) < \infty, \quad t \in \mathbb{R}_+, j \in \mathbb{N}.$$

Assume further that $\kappa(\eta, s, j, z) = \kappa_j(\eta, s)z$, for some predictable processes κ_j, $j \in \mathbb{N}$. For any $h \in \mathcal{A}$ we then have

$$\int h(\eta, s, j)\zeta(\mathrm{d}(s, j)) = \sum_{j\in\mathbb{N}} \int h(\eta, s, j)\kappa_j(\eta, s)\mathrm{d}\zeta_j(s) \quad \mathbb{P}\text{-a.s.},$$

where $\zeta_j(t) := \iint \{s \le t\} z\hat{\eta}(\mathrm{d}s \times \{j\} \times \mathrm{d}z)$, $t \ge 0$, are independent square integrable processes with independent increments and mean 0 (and no fixed jumps). Assume now moreover, that $\lambda_j(\mathrm{d}(s, z)) = \mathrm{d}s\nu_j(\mathrm{d}z)$ for measures ν_j on $\mathbb{R}^*$, so that the ζ_j are square integrable Lévy martingales. Then we can choose $J_j(s, \mathrm{d}z) = \nu_j(\mathrm{d}z)$ and (5.14) simplifies to

$$h_f(\eta, s, j) = \kappa(\eta, s, j)^{\oplus} \left(\int z^2 \nu_j(\mathrm{d}z)\right)^{-1} \int z\mathbb{E}[D_{(s,j,z)} f(\eta)|\eta_{s-}]\nu_j(\mathrm{d}z). \tag{5.15}$$

This is the main result in [2]. In fact, the model in [2] allows the processes ζ_j to have a Brownian component but considers only finitely many non-zero measures ν_j.

Remark 5.7. The result of this and the previous section can probably be generalized so as to cover L^2-valued signed random measures driven by an independent random measure with Gaussian component. This would require to replace the pathwise defined difference operator by a suitably defined Malliavin derivative. In fact, Øksendal and Proske [26] extended the results of [2] (see Example 5.6) to general square integrable Lévy processes by establishing an appropriate version of Theorem 3.1. In this paper we make no attempt to extend our Theorem 3.1 beyond the Poisson setting.

Remark 5.8. While this paper has been under review we became aware of the recent work [6] that also studies the minimal variance hedging problem in a financial market with a continuum of assets. The results in [6] allow for a Gaussian component and are based on a version of Theorem 3.1, where the derivative is defined via a limit procedure. As a consequence, the resulting formulas for the minimal variance hedge are less explicit than our Theorem 5.4.

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